Boundary conditions and defect lines in the Abelian sandpile model

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We add a defect line of dissipation, or crack, to the Abelian sandpile model. We find that the defect line renormalizes to separate the two-dimensional plane into two half planes with open boundary conditions. We also show that varying the amount of dissipation at a boundary of the Abelian sandpile model does not affect the universality class of the boundary condition. We demonstrate that a universal coefficient associated with height probabilities near the defect can be used to classify boundary conditions.

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I. INTRODUCTION

The Abelian sandpile model (ASM) was introduced by Bak, Tang, and Wiesenfeld as a model of self-organized criticality [1]. This well-known model was designed to demonstrate how simple rules can drive a system to a critical point, and thus produce power laws, without any fine tuning of parameters. It has thus been used to explain power laws in a wide range of systems—see Refs. [2,3] for a review. Since the ASM was first introduced, a number of variations on the model have been introduced—see Ref. [4] for a review. However, the original ASM still provides a simple, important, and robust model for the generation of power laws.

The ASM is defined on a square lattice. Each site a of the lattice has a height variable, h_a, which can be any integer from 1 to 4, inclusive, where h_a represents the number of grains of sand at that site. At each time step, a grain of sand is added to a random site of the lattice. After the addition of the grain, any site with more than four grains of sand is unstable, and collapses, losing four grains of sand, while each of its four neighbors gains one grain of sand. Unstable sites are repeatedly collapsed, until every site is stable—i.e., no site has more than four grains. Then, the next time step, another grain is added, and the entire process is repeated.

The original ASM is spatially homogenous (except for the boundaries, which break translational invariance), and most modifications of the sandpile model have kept this feature. However, here we consider the effects of a crack, represented by a defect line, along which grains of sand can be lost; in other words, along which the number of grains is not conserved. In previous studies, dissipation was added to the bulk of the ASM (not breaking translational invariance), and was shown to take the ASM off its critical point [5–7]. Our defect line of dissipation breaks translational invariance, and we show that it causes the two-dimensional plane of the ASM to renormalize into two half planes with open boundary conditions. This shows that cracks in the ASM are highly relevant, and essentially cleave the sandpile into separate pieces. We demonstrate this by looking at the universal coefficient associated with the modification of unit height probabilities at large distances from the defect, and at the correlation function between unit height variables on opposite sides of the defect. The Green function for an ASM with a defect line is calculated in Sec. III, and results for the height probabilities and correlations are presented in Sec. IV.

For most models of interest in condensed matter physics, the bulk properties can be studied with the boundary playing little or no role. For example, the two-dimensional Ising model is often studied on a torus, so as to eliminate boundary effects. However, this is not possible for the ASM. In the bulk of the ASM, the number of grains of sand is conserved during each toppling. If this was true for all sites, then eventually we would reach a state where topplings continued without end. The ASM thus needs sites with dissipation—that is, sites where the number of grains is not conserved. The most natural way to do this is with open boundary conditions; sites at the open boundaries become unstable when they have more then four grains (just as in the bulk), but have only three neighbors to send grains to, and send the fourth grain "off the edge," removing it from the system. Since this dissipation is necessary for a well-defined sandpile, the boundary plays a crucial role in the ASM, even when we are focused at points in the bulk. Correlation functions far from the boundary are independent of the boundary conditions, just as in other condensed matter statistical mechanical models; but the presence of dissipation somewhere in the ASM (e.g., at the open boundary) is necessary for the model to be well defined.

We consider the effects of varying amounts of dissipation along a boundary, and show that any amount of dissipation at the edge results in the open boundary universality class. The Green function is calculated in Sec. V, and results for the height probabilities and correlations are presented in Sec. VI.

These results are intuitively reasonable, since dissipation should be relevant in regions of the ASM where the particles have no other way to leave the ASM. However, it was also possible that such modifications could have resulted in new, as yet undiscovered, boundary conditions or defect states. For example, Bariev, and McCoy and Perk, added a line defect of modified bond strengths to the Ising model, and found that they were able to continuously vary the dimension of the spin operator along the defect by varying the defect bond strength [8,9]. This continual variation occurred despite the fact that the Ising model only has three conformally invariant boundary conditions.

The ASM has been associated with a conformal field theory (CFT). While CFT's are generally well understood,

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the ASM is a logarithmic CFT (LCFT) (specifically, the $c=-2$ CFT), many aspects of which are still not well understood [10]. In particular, our understanding of the boundary states of LCFT's is still fragmentary, and recent results on boundaries of the $c=-2$ CFT have been partially contradictory [11–15]. Connections between the LCFT boundary states, and the ASM boundary states were made in Ref. [16], but the ASM representation of some of the $c=-2$ LCFT boundary states is still unknown. Modifications to the ASM such as those described in this paper, and searches for other boundary conditions, could help eludicate these relationships. Our results provide some evidence that the open and closed boundary conditions are the only possibilities for the ASM, although it is still possible that further calculations in this vein could uncover new boundary conditions.

The identification of boundary states as closed, or open, or in some new, as yet undiscovered class, uses arguments from CFT that the coefficients associated with the falloff of expectation values (height probabilities) at large distances from the defect should be universal. Our results both use this expected universality, and confirm it, since we find, for example, that the coefficient is unaffected by varying a free parameter corresponding to the amount of dissipation at the boundary. This confirmation, while expected, is valuable, given the anomalous and unsettled nature of boundary LCFT associated with the ASM. This is a particularly important point in light of recent arguments that use this universality to argue that the four height variables in the ASM must correspond with different fields in the corresponding CFT [17].

II. THE FORMALISM

Dhar pointed out that ASM is highly analytically tractable because of its Abelian nature—the same state results whether grains of sand are added first at site a and then at site b, or first at site b and then at site a [18]. This is the basis of a well-established formalism for analyzing the ASM—see Ref. [19] for a review. We only give a quick coverage of the essential points here.

It is useful to first generalize the above description of the ASM, to allow for more complicated topplings. The dynamics of the model are described by a toppling matrix Δ_{ab} where a and b label sites of the lattice. The dimension of Δ is equal to the number of sites in the lattice, so Δ becomes infinite dimensional as the size of the lattice goes to infinity. We say that site a is unstable if its height h_a is greater than Δ_{aa}. If site a is unstable, then every height changes by $h_b \rightarrow h_b - \Delta_{ab}$ (including at the site $b=a$). We have the standard ASM, with open boundary conditions, if Δ_{ab} is 4 when $a=b$, -1 when a and b are nearest neighbors, and 0 otherwise.

Dhar showed that the states of any sandpile, given certain general conditions on the form of Δ, are divided into transient states, which occur with probability zero after long amounts of time, and recurrent states, which all occur with equal probability after long amounts of time. The number of possible recurrent configurations is given by $\det(\Delta)$ [18].

Furthermore, Majumdar and Dhar also showed how to find the probability for a site to have height 1, and the joint probability for two sites to both have height 1 (as well as other, more complicated probabilities) [20,21]. The toppling matrix is modified by removing specific bonds, and changing the toppling condition at certain sites. For example, if we want to force site a to have height 1, we change the toppling matrix so that $\Delta_{aa}=1$, and remove three of the bonds to neighboring sites (setting $\Delta_{ab}=\Delta_{ba}=0$ for those bonds). With this modified toppling matrix Δ' site a is now guaranteed to have height 1, and $\det(\Delta')$ gives the number of recurrent configurations with $h_a=1$. While Δ and Δ' are infinite-dimensional matrices (for an infinite lattice), $B \equiv \Delta' - \Delta$ is 0 outside of a 4×4 submatrix. So $\det(\Delta')/\det(\Delta) = \det(1 + B\Delta^{-1})$ is an easily computable 4×4 matrix determinant, which gives the probability that, in a randomly chosen recurrent configuration, the site a will have height 1. The same process, with a different (8×8) matrix B, can be used to find two-point correlations of height 1 variables.

This process requires us to calculate the Green function $G \equiv \Delta^{-1}$. The Green function has long been known for the standard ASM, where Δ is simply the lattice Laplacian [22]. However, in the following sections we will be dealing with different toppling conditions, and so will need to calculate the Green function for these new Δ's.

III. GREEN FUNCTION FOR THE DEFECT LINE

We introduce a defect line (or crack) through the middle of the ASM, allowing dissipation to take place along the defect, and not just along the open boundary conditions. We take the lattice to be size $M\times(2L-1)$, with the x dimension taking on the values $i=0,1,\ldots,(M-1)$, and the y dimension taking on the values $j=-(L-1),-(L-2),\ldots,(L-2),(L-1)$. We take open boundary conditions along all edges, and put the defect along the line $j=0$. Along this line, the height variable can take on the values $1,2,\ldots,(4+m)$, where $m>0$. A site along the defect topples if its height is greater than $(4+m)$. When it topples, it sends one grain to each of its four neighbors, and m grains of sand are dissipated (i.e., disappear from the sandpile).

When m is a positive integer, the theory has its most obvious physical interpretation, but the theory can be modified to give a sensible interpretation for any rational, positive, value of m [5]. If in each toppling, c_1 grains are toppled, and c_2 grains sent to each neighbor, where c_1 and c_2 are integers, then the ratio of grains dissipated to grains moved $m/4 \leftrightarrow [c_1/(4c_2)]-1$ can be any rational integer.

The toppling matrix Δ is the same as for the standard ASM, except that $\Delta_{aa}=4+m$ for sites a along the defect. When $m=0$ it becomes the standard ASM. The toppling matrix can be written as

$$\Delta_{(i,j),(i',j')} = \delta_{ii'}\Delta^{(2)}_{jj'} + \delta_{jj'}\Delta^{(1)}_{ii'}, \tag{1}$$

with

$$\Delta^{(1)}_{ii'} \equiv \begin{cases} 2 & \text{if } i = i' \\ -1 & \text{if } i = i' \pm 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$\Delta^{(2)}_{jj'} \equiv \begin{cases} 2 & \text{if j} = j' \neq 0 \\ \text{m} + 2 & \text{if j} = j' = 0 \\ -1 & \text{if j} = j' \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Since Δ is Hermitian, if we find all of its normalized eigenvectors, we can easily invert it. Suppose that the eigenvectors of Δ are $e_{\vec{p},\vec{x}}$, with eigenvalues $\lambda_{\vec{p}}$. $\vec{p}$ and $\vec{x}$ are two-dimensional vectors and the number of possible values of $\vec{p}$ is equal to the dimension of Δ, which is in turn equal to the number of sites in the lattice. Then

$$G_{\vec{x},\vec{y}} = \Delta^{-1}_{\vec{x},\vec{y}} \tag{4}$$

$$= \sum_{\vec{p}} \frac{1}{\lambda_{\vec{p}}} e_{\vec{p},\vec{x}} e_{\vec{p},\vec{y}}. \tag{5}$$

The form of Δ in Eq. (1) implies that the eigenvectors of Δ factorize into eigenvectors of $\Delta^{(1)}$ and $\Delta^{(2)}$.

We thus want the eigenvectors of $\Delta^{(2)}$. (The eigenvectors of $\Delta^{(1)}$ are not only simpler, but immediately follow from the eigenvectors of $\Delta^{(2)}$, by setting $m=0$.) j and j' range from $-(L-1)$ to $(L-1)$, so $\Delta^{(2)}$ has $2L-1$ eigenvectors. The eigenvectors fall in three classes. There are $(L-1)$ oscillatory eigenvectors that are antisymmetric about $j=0$, and have momenta p evenly spaced between 0 and π [$p=n\pi/L$, $n \in \mathcal{Z}$, $1 \leqslant n \leqslant (L-1)$]. There are another $(L-1)$ oscillatory eigenvectors that are symmetric about $j=0$, and have momenta p in the range $0<p<\pi$, where the p solve a transcendental equation; in the limit $L \to \infty$ these momenta p also become equally spaced between 0 and π. Finally, there is one exponentially decaying eigenvector, symmetric about $j=0$.

Since Δ is Hermitian, we can immediately obtain its inverse from these eigenvectors. The sums over the two oscillatory sets of eigenvectors each produce integrals in the limit $L \to \infty$, $M \to \infty$, using the Euler-MacLaurin formula. The last, exponentially decaying, eigenvector produces a single, discrete contribution to the Green function. Writing the Green function as a sum of the contributions from the three classes of eigenvectors gives

$$G(i,j,i',j') = \sum_{a=1}^{3} G^{(a)}(i,j,i',j'), \tag{6}$$

$$G^{(1)}(i,j,i',j') = \tfrac{1}{2} G_0(i-i',j-j') - \tfrac{1}{2} G_0(i-i',j+j'), \tag{7}$$

$$G^{(2)}(i,j,i',j') = \tfrac{1}{2} G_0(i-i',|j|-|j'|) + \delta G^{(2)}(i-i',|j|+|j'|), \tag{8}$$

$$G^{(3)}(i,j,i',j') = \frac{(-1)^{j+j'}}{2\sqrt{k^2-1}} \frac{m}{\sqrt{m^2+4}} (K-\sqrt{K^2-1})^{|i-i'|} \times \left(-\frac{m}{2} - \frac{\sqrt{m^2+4}}{2} \right)^{-(|j|+|j'|)}, \tag{9}$$

where we have defined

$$K \equiv 2 + \tfrac{1}{2}\sqrt{m^2+4}. \tag{10}$$

G_0 is the well-known bulk Green function [22]

$$G_0(i,j) \equiv \int_0^{2\pi} \frac{dp_i}{2\pi} \int_0^{2\pi} \frac{dp_j}{2\pi} \frac{\cos(p_i i)\cos(p_j j) - 1}{4 - 2\cos p_i - 2\cos p_j}. \tag{11}$$

We have also defined

$$\delta G^{(2)}(i,j) \equiv \left(1 - \frac{m^2}{2}\right) G^{(2a)}(i,j) + m[G^{(2a)}(i,j-1) - G^{(2a)}(i,j+1)] - \frac{1}{2}[G^{(2a)}(i,j-2) + G^{(2a)}(i,j+2)] + c_m, \tag{12}$$

$$G^{(2a)}(i,j) \equiv \int_0^{2\pi} \frac{dp_i}{2\pi} \int_0^{2\pi} \frac{dp_j}{2\pi} \frac{\cos(p_i i)\cos(p_j j) - 1}{4 - 2\cos p_i - 2\cos p_j} \times \frac{1}{m^2 + 4\sin^2 p_j}, \tag{13}$$

$$c_m \equiv \int_0^{2\pi} \frac{dp_i}{2\pi} \int_0^{2\pi} \frac{dp_j}{2\pi} \frac{1}{2 - \cos p_i - \cos p_j} \frac{\sin^2 p_j}{m^2 + 4\sin^2 p_j}. \tag{14}$$

We want the behavior of $G(i,j,i,j')$ for $|j|+|j'|$ large. The expansion of the bulk Green function $G_0(0,j)$ for large j is well known [22]:

$$G_0(0,j) \to -\frac{1}{2\pi}\ln(j) - \frac{1}{\pi}\left(\frac{\gamma}{2} + \frac{3}{4}\ln 2\right) + \frac{1}{24\pi j^2} + \cdots, \tag{15}$$

where $\gamma=0.577\ldots$ is the Euler-Mascheroni constant. We also need the behavior of $\delta G^{(2a)}(0,j)$ for j large. The integral over p_j in Eq. (13) can be done exactly, and making the substitution $z=e^{ip_i}$ gives a contour integral around the unit circle. The integrand has two poles inside the unit circle, but these give contributions which either decay exponentially with j, or are independent of j, neither of which affects our height calculations; so these contributions can be dropped. The algebraic j dependence comes from the branch cut in the integrand, running from $z=3-\sqrt{8}$ to $z=3+\sqrt{8}$, which gives

$$G^{(2a)}(0,j) \to \frac{1}{\pi} \mathcal{P} \int_{3-\sqrt{8}}^{1} dz \frac{z^j - 1}{z-1} f(z), \tag{16}$$

$$f(z) \equiv \frac{1}{\sqrt{-z^2+6z-1}} \left(\frac{z^2}{(z^2-1)^2 - m^2 z^2} \right), \tag{17}$$

where $\mathcal{P}$ indicates that we take the principal part of the integral. We can use this to find the behavior of $G^{(2a)}(0,j)$ for large j, by separating out the contributions from z near 1, and expanding in a Laurent series in j. This then gives the expansion of $\delta G^{(2)}(0,j)$,

$$\delta G^{(2)}(0,j) = \frac{1}{4\pi} \ln j + \frac{1}{m\pi j} + \frac{m^2 - 96}{48\pi m^2 j^2} + \cdots, \quad (18)$$

dropping terms independent of j.

IV. HEIGHT PROBABILITIES FOR THE DEFECT LINE

Now that we have the Green function for the defect line, we can use it to calculate the height correlations, using the methods outlined in Sec. II. We find that the probability for a site a distance j from the defect to have height 1 is

$$\mathrm{Prob}(h_{(i,j)} = 1) = \frac{2(\pi - 2)}{\pi^3}\left(1 + \frac{1}{4j^2} - \frac{1}{2mj^3} + \cdots\right). \quad (19)$$

The constant term $2(\pi-2)/\pi^3$ is the bulk probability for a site to have height 1, first calculated in Ref. [20]. It was also found in Ref. [20] that the correlation function between two height 1 operators is

$$\mathrm{Prob}(h_{(i_1,j_1)} = h_{(i_2,j_2)} = 1) = \left(\frac{2(\pi - 2)}{\pi^3}\right)^2\left(1 - \frac{1}{2r^4} + \cdots\right), \quad (20)$$

where

$$r \equiv \sqrt{(i_1 - i_2)^2 + (j_1 - j_2)^2}. \quad (21)$$

Thus, the height 1 operator is a dimension 2 operator. Based on the identification of the ASM as a conformal field theory [21,23], the coefficient of $1/j^2$, in expectation values of dimension 2 operators a distance j from a boundary, is expected to be a universal number characteristic of the boundary condition [24]. And, in fact, this coefficient of $+1/4$ in Eq. (19) is exactly the coefficient seen for the height 1 probability at large distances from an open boundary condition, as shown by Brankov, Ivashkevich, and Priezzhev [25]. This indicates that, upon renormalization, the defect line becomes an open boundary.

It is only sensible to talk about conventional boundary conditions at the defect line, if the two half planes on either side of the defect have somehow been separated. Evidence that the defect renormalizes to separate the half planes can be seen by looking at correlation functions of points on opposite sides of the defect. If, upon renormalizing the defect, the two sides of the defect were still "connected," we would expect that height variables on opposite sides would still fall off as $1/r^4$, since the height 1 operator has dimension 2. (Calculations of correlation functions along boundaries by Ivashkevich have shown that the height one operator also has dimension 2 along open boundaries [26].) However, we find that

$$\mathrm{Prob}(h_{(i,j)} = 1, h_{(i,-j)} = 1) - \mathrm{Prob}(h_{(i,j)} = 1)\mathrm{Prob}(h_{(i,-j)} = 1) = \left(\frac{2(\pi - 2)}{\pi^3}\right)^2\left[-\frac{1}{8m^2 j^6} + O\left(\frac{1}{j^7}\right)\right]. \quad (22)$$

While the height variable is a dimension 2 operator, its correlations across the defect fall off as $1/r^6$. The coefficient of the $1/r^4$ term renormalizes to zero, and the $1/r^6$ term is nonuniversal, depending continuously on m. We thus conclude that the defect renormalizes to generate two separate half planes with open boundary conditions.

This is physically reasonable. Adding dissipation throughout the bulk of the ASM is known to be relevant, driving the system off criticality [5–7]. More recently, adding dissipation in the bulk was identified with adding the integral of a dimension 0 variable (the logarithmic partner of the identity) throughout the bulk [23]. It would thus appear that the local addition of dissipation should be represented by a dimension 0 operator, which would mean that adding dissipation along a defect line should be relevant, as we have found here. In short, cracks in the ASM cleave the plane into disconnected regions with open boundary conditions. Similar separation with relevant perturbations along a defect occurs in other models—see, for example, Ref. [27]. However, these results were not inevitable; as already noted, line defects added by Bariev, and McCoy and Perk, to the Ising model, resulted in a continual range of defect lines, along which the dimension of the spin operator could be continuously varied, despite the fact that the Ising model only has three possible boundary conditions.

V. GREEN FUNCTION FOR THE MODIFIED BOUNDARY CONDITIONS

The identification of a line of dissipation with open boundary conditions brings up the question of whether any other universality classes of boundary conditions with dissipation are even possible. The open and closed boundary conditions are the most natural to impose on the ASM, but other boundary conditions than these two conventional ones can be written down. We create new boundary conditions by varying the amount of dissipation along the boundary, and show that, regardless of the amount of dissipation, we stay in the open boundary universality class (so long as the amount of dissipation is nonzero—that is, so long as the boundary is not closed).

It is convenient to change the dimensions of the lattice from those of Sec. III. We take the lattice to be of size $M \times L$, with the x dimension taking on the values $i = 0, 1, \ldots, (M-1)$, and the y dimension taking on the values $j = 0, 1, \ldots, (L-1)$. We impose a modified boundary condition along $j=0$, and open boundary conditions on the other three edges. (In the end, we take the limits $L \to \infty$ and $M \to \infty$, so our results should anyway be insensitive to the boundary conditions on these three edges.)

We allow the height variable on the boundary $j=0$ to take on values $1, 2, \ldots, b$. Sites on the boundary become unstable when their height is greater than b, at which point they topple, giving one grain to each of their three neighbors, and dropping $b-3$ grains off the edge. For $b=3$ this is the closed boundary condition, and for $b=4$ this is the open boundary condition. For $b<3$, sand is generated with each toppling, rather than dissipated, creating the possibility of never-ending cycles of toppling. We therefore only consider $b>3$. The system can be given a sensible interpretation for any rational value of $b \geqslant 3$ [5]. The toppling matrix between sites (i,j) and (i',j') can be written as

$$\Delta_{(i,j),(i',j')} = \delta_{ii'}\Delta^{(3)}_{jj'} + \delta_{jj'}\Delta^{(1)}_{ii'}, \tag{23}$$

where $\Delta^{(1)}$ was defined in Eq. (2), and

$$\Delta^{(3)}_{jj'} \equiv \begin{cases} 2 & \text{if } j = j' \neq 0 \\ b-2 & \text{if } j = j' = 0 \\ -1 & \text{if } j = j' \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

As with the defect, if we can find all the eigenvectors of Δ, we can easily invert it. $\Delta^{(3)}$, being L dimensional, has L eigenvectors. When $3<b<5$, $\Delta^{(3)}$ has L eigenvectors that are oscillatory functions of j, with momenta p, which satisfy a transcendental equation. In the limit $L\to\infty$, the momenta are evenly spaced over the range $0<p<\pi$. When $b>5$, $\Delta^{(3)}$ only has $(L-1)$ such oscillatory eigenvectors, and one last eigenvector that is exponentially decaying in j.

In the Green function, the summation over oscillatory eigenvectors can be turned into an integral in the limit $L\to\infty$, $M\to\infty$, with the Euler-MacLaurin formula. For $b>5$, the single, exponentially decaying eigenvector produces a separate, discrete contribution to the Green function. The Green function is then given by

$$G(i,j,i',j') = \tilde{G}(i,j,i',j') + \theta(b-5)G^{\exp}(i,j,i',j'), \tag{25}$$

where

$$\begin{aligned}\tilde{G}(i,j,i',j') \equiv & \int_0^{2\pi}\frac{dp_i}{2\pi}\int_0^{2\pi}\frac{dp_j}{2\pi}\frac{\cos[p_i(i-i')]}{2-\cos p_i-\cos p_j} \\ & \times\frac{1}{(b-4)^2+1+2(b-4)\cos p_j} \\ & \times\{(b-4)\sin(p_j j)+\sin[p_j(j+1)]\} \\ & \times\{(b-4)\sin(p_j j')+\sin[p_j(j'+1)]\},\end{aligned} \tag{26}$$

and

$$G^{\exp}(i,j,i',j') \equiv \frac{(b-3)(b-5)}{2\sqrt{k^2-1}}(4-b)^{-j-j'-2}(k-\sqrt{k^2-1})^{|i-i'|}. \tag{27}$$

We have defined

$$k \equiv 1+\frac{(b-3)^2}{2(b-4)}, \tag{28}$$

and

$$\theta(x) \equiv \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x\leqslant 0. \end{cases} \tag{29}$$

At first sight, this equation for the Green function seems to indicate that G has a slope discontinuity at $b=5$. However, this is not the case. $\tilde{G}$ is not smooth at $b=5$, and expanding $\tilde{G}$ as a function of b near $b=5$ shows that the combination $\tilde{G}+\theta(b-5)G^{\exp}$ is actually smooth to all powers of $(b-5)$. Inspection shows that G is also a smooth function of b for all other b in the range $3<b<\infty$ (including $b=4$).

We need the Green function in two limits. First, for $i=i'$ and $j+j'$ large, it is useful to write

$$\begin{aligned}\tilde{G}(i,j,i',j') = & G_0(i-i',j-j') - G_0(i-i',j+j') \\ & + \delta G(i-i',j+j'),\end{aligned} \tag{30}$$

$$\begin{aligned}\delta G(i,j) \equiv & \int_0^{2\pi}\frac{dp_i}{2\pi}\int_0^{2\pi}\frac{dp_j}{2\pi}\frac{\cos(p_i i)}{2-\cos p_i-\cos p_j} \\ & \times\frac{\sin p_j}{(b-4)^2+1+2(b-4)\cos p_j} \\ & \times[\sin p_j\cos(p_j j)+(b-4+\cos p_j)\sin(p_j j)].\end{aligned} \tag{31}$$

In $\delta G(0,j)$, we can do the integral over p_i exactly, and then set $z=e^{ip_j}$. As before, the main contribution comes from the branch cut between $z=3-\sqrt{8}$ and $z=1$. Expanding the integral near $z=1$ gives

$$\delta G(0,j) \approx \frac{1}{\pi(b-3)j} - \frac{1}{\pi(b-3)^2 j^2} + \frac{b^2-8b+19}{2\pi(b-3)^3 j^3} + \cdots. \tag{32}$$

We also need the expansion of the Green function along the defect—that is, for $j=j'=0$ and $|i-i'|\gg 0$. Using similar methods as before, we find

$$\begin{aligned}\tilde{G}(x=|i-i'|, j=j'=0) \approx & \frac{1}{\pi(b-3)^2 x^2} \\ & - \frac{b^2-18b+57}{2\pi(b-4)^4 x^4} + O\left(\frac{1}{x^6}\right).\end{aligned} \tag{33}$$

VI. HEIGHT PROBABILITIES FOR MODIFIED BOUNDARY CONDITIONS

Using the Green function for modified boundary conditions, we can calculate unit height probabilities with the methods described in Sec. II. We find that the probability for a site a distance j from the boundary to have height 1 is

$$\text{Prob}(h_{(i,j)}=1) = \frac{2(\pi-2)}{\pi^3}\left(1+\frac{1}{4j^2}-\frac{1}{2(b-3)}\frac{1}{j^3}+\cdots\right). \tag{34}$$

As discussed earlier, the coefficient of the $1/j^2$ term is expected to be a universal number characteristic of the boundary condition [24], and is equal to $+1/4$ for the open boundary condition [25]. We see here that the coefficient is $+1/4$, and independent of b for $b>3$. This both confirms the expectation that the coefficient should be universal, and indicates that the boundary is in the open boundary universality class for any amount of dissipation ($b>3$).

Note that the coefficient of $1/j^3$ is nonuniversal, and diverges as $b\to 3$, indicating that $b=3$ is a special point as we vary b. $b=3$ corresponds to the closed boundary condition,

and it is already known that the coefficient of the $1/j^2$ term is different $(-1/4)$ for the closed boundary conditions; this is appropriate, since the closed and open boundary conditions are clearly in different universality classes [25].

Boundary correlations along the $j=0$ boundary can be calculated, and contain no surprises. The correlation function between sites $(i,0)$ and $(i',0)$ falls off as $1/|i-i'|^4$ for all values of b:

$$\begin{aligned}\mathrm{Prob}(h_{(i,0)}=1,h_{(i',0)}=1)-\mathrm{Prob}(h_{(i,0)}=1)\mathrm{Prob}(h_{(i',0)}=1)\\ =-\left(\frac{(1-c_0+c_2)(-1+3c_0-4c_1+c_2)}{(b-3)\pi}\right)^2\frac{1}{|i-i'|^4}\\ +O\left(\frac{1}{|i-i'|^6}\right).\end{aligned} \tag{35}$$

We have defined $c_x \equiv G(x=|i-i'|, j=j'=0)$. Equations (25)–(27) can be used to find analytic expressions for c_x, for $x=0,1,2$. However, the expressions are long and not particularly enlightening, so are not presented here. The (absolute value of the) coefficient of the $1/|i-i'|^4$ term is plotted in Fig. 1. It falls off smoothly with increasing b.

The coefficient of $1/|i-i'|^4$ in Eq. (35) diverges as b approaches 3, reflecting the fact that $b=3$ is a fixed point of the renormalization group (RG) flows, leading to nonsmooth behavior in physical properties. However, the Green function, and height correlations calculated from it, are perfectly smooth as we vary b through 4. It would appear that the RG flows take us from $b=3$ to $b=\infty$, and that $b=4$ is not a fixed point of the RG flows. However, $b=\infty$ is in a sense the same as $b=4$, in that both equally well represent the open boundary condition; if $b=\infty$, the sites $j=0$ can hold an infinite number of grains, and never topple—the sandpile thus acts as if $j=1$ was the boundary, with an open boundary condition, where grains fall "off the edge" to $j=0$.

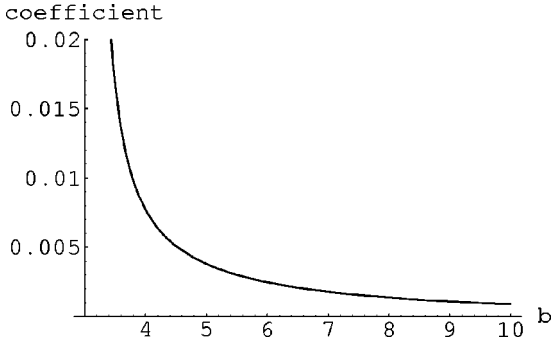

FIG. 1. Coefficient of the $1/r^4$ term in the two-point correlation function.

We have shown that the addition of dissipation along a defect line separates the ASM into two half planes, each with open boundary conditions. This brought up the question of whether there are other universality classes of boundary conditions, with varying amounts of dissipation along the boundary. We find that any amount of dissipation along a boundary results in the open boundary universality class at large distances. Classes of boundary conditions were identified by the universal coefficient of the unit height probability, far from the boundary or defect.

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